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ON THE VALIDITY OF WAGNER HYPOTHESIS

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Abstract—The Wagner hypothesis has been widely accepted in the buckling analysis of thin-walled members. Recently, the validity of Wagner hypothesis has been questioned by Ojalvo based on the theorem of minimum potential energy following the method used by Bleich. Since then, efforts have been made by several investigators to settle the dispute on the validity of Wagner hypothesis, but no general agreement has yet been reached. The purpose of this paper is to show more clearly the validity of Wagner hypothesis using the same method as adopted by Ojalvo. In this way, we show the difference between the two formulations and point out more precisely why Ojalvo's theories are incorrect.

1. INTRODUCTION

The "Wagner hypothesis" (1936) has generally been accepted in the buckling analysis of thin-walled members. However, Ojalvo (1981) rejected this hypothesis and presented a new theory for the flexural torsional buckling of columns and monosymmetric beams. Soon after the presentation of the new theory, critical discussions (Kitipornchai and Dux, 1982; Leko, 1982; Trahair, 1982; Studnicka and Kristek, 1982; Haaijer, 1983) were made against his new theory. In spite of these discussions, Ojalvo (1982, 1983, 1987a, b) has been repeatedly insisting on the validity of his theory up to the present. Recently, in order to validate his theory, Ojalvo (1987a, b) showed a different procedure based on the theorem of minimum potential energy following the method shown by Bleich (1952). By this procedure he obtained exactly the same theory as he derived from the analysis of free body diagram (Ojalvo, 1981) and again rejected the Wagner hypothesis. This rejection was also argued to be invalid by Kitipornchai *et al.* (1987) and Trahair and Papangelis (1987). They confirmed the validity of the Wagner effects by analyzing the thin-walled beams as a three-dimensional solid and then using this to obtain a one-dimensional theory. Although not well known in North America, a more refined and exact method similar to the above has already been used in Japan (Nishino *et al.*, 1973; Goto *et al.*, 1985). Nishino *et al.* (1973) derived a refined finite displacement theory of thin walled members from continuum mechanics, utilizing the theorem of virtual work under beam assumptions. More recently, Goto *et al.* (1985) obtained an exact finite displacement theory of rods with solid cross section, where no approximations are introduced except the usual beam assumptions. Both of these theories yield Wagner's K term naturally and automatically through variational calculus without introducing the Wagner hypothesis.

In spite of the efforts so far to settle the dispute on the validity of Wagner hypothesis, an agreement has not yet been obtained. This is mainly because the two conflicting theories, i.e. theories with and without Wagner hypothesis, use different procedures to validate themselves.

The purpose of this paper is to show more clearly the validity of Wagner hypothesis. To this end, the same method as Ojalvo adopted recently, that is, the method based on the theorem of minimum potential energy, is utilized to obtain a buckling theory of thin-walled members. This procedure enables us to point out the Ojalvo's mistakes more precisely by showing the discrepancy between his formulation and ours. Herein, as analyzed by Ojalvo (1981, 1987b), the flexural torsional buckling of a column as well as that of a monosymmetric beam are chosen as examples.

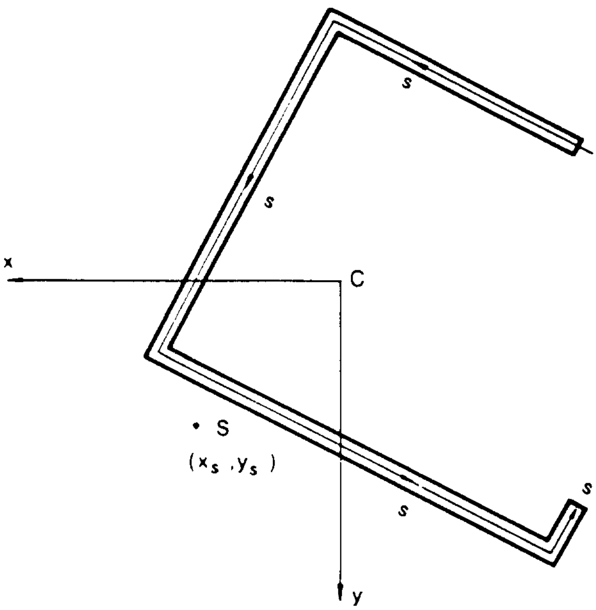

Fig. 1. Cross section of thin-walled member.

2. COORDINATES AND DISPLACEMENT FIELD

Consider a straight, thin-walled member as shown in Fig. 1. A Cartesian coordinate system (x, y, z) is introduced at the initial configuration of the member with the coordinate z along the member axis. Coordinate axes (x, y) are chosen such that they coincide with the principal axes of the cross section. The member axis at the origin of the (x, y) coordinates is specifically called centroidal axis. In addition to the (x, y, z) coordinates, coordinate s is introduced along the middle line of the cross section.

Translational displacement components in the directions of (x, y, z) coordinates are expressed by (u, v, w).

From the condition of no change of cross-sectional shapes, the x and y components of translational displacement on the cross section can be expressed as

$$u = u_s - (y - y_s)\theta, \quad v = v_s + (x - x_s)\theta \tag{1a, b}$$

where θ is the rotational angle around the z axis and subscript s denotes a quantity on the shear center $S(x_s, y_s)$ of the cross section. Further, utilizing the Bernoulli–Euler hypothesis, the displacement in z-direction can be obtained as

$$w = w_c - xu_s' - yv_s' - \omega\theta' \tag{2}$$

where $(\cdot)'$ denotes differentiation with respect to z, subscript c expresses a quantity on the centroidal axis, and $\omega(s)$ is a warping function. In the buckling analysis, it is necessary to distinguish the incremental displacements at buckling from the total displacements up to buckling. Hence, for this distinction, the quantities with bars and stars shown below are used throughout this paper.

$\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{\theta}$ (total displacements up to buckling)

u^*, v^*, w^*, θ^* (incremental displacements at buckling).

For later convenience, the increment of the translational displacements (u^*, v^*, w^*) are expressed in terms of longitudinal displacement w_c^* at centroid and the transverse displacements (u_s^*, v_s^*) at shear center. These expressions are obtained from eqns (1) and (2) as

$$u^* = u_s^* - (y - y_s)\theta^*, \quad v^* = v_s^* + (x - x_s)\theta^*$$
$$w^* = w_c^* - xu_s^{*\prime} - yv_s^{*\prime} - \omega\theta^{*\prime}. \tag{3a–c}$$

For the ease of mathematical manipulations, the origin of coordinate s can be selected in such a manner that

$$\int_A \omega \, dA = 0 \tag{4}$$

in which $\int_A(\cdot)\, dA$ denotes an integration over the cross-sectional area.

Besides eqn (4), the coordinates and the displacement components adopted here yield the following convenient relations.

Coordinate axes x and y coincide with the principal axes of the cross section. Thus, we have

$$\int_A x \, dA = 0, \quad \int_A y \, dA = 0, \quad \int_A xy \, dA = 0. \tag{5a–c}$$

Point S is the shear center of the cross section for which the following relation holds.

$$\int_A \omega x \, dA = 0, \quad \int_A \omega y \, dA = 0. \tag{6a, b}$$

3. TOTAL POTENTIAL ENERGY

The total potential energy Π of a structural system is composed of the internal strain energy U and the potential energy W of the external loads and is expressed by

$$\Pi = U + W. \tag{7}$$

When deriving the value of the total potential energy in buckling analysis, the load state just before buckling is used as a reference state. In the derivation of total potential energy, we have to consider incremental displacement terms at least up to the second order since the buckling load is known from the second variation ($\delta^2\Pi$) of Π. Equilibrium equations can be obtained from the following stationary condition.

$$\delta\Pi = \delta U + \delta W = 0. \tag{8}$$

4. BUCKLING OF A CENTRALLY LOADED COLUMN

4.1. *Buckling phenomena*

Consider a thin-walled column of length L subjected to an end compressive force $\bar{P}$ passing through centroid.

Up to the critical compressive force $\bar{P}_{cr}$, this column is strained only in the longitudinal direction. The buckling of this structure is classified as a symmetric bifurcation (Tompson and Hunt, 1973). Thus, when the compressive force $\bar{P}$ reaches $\bar{P}_{cr}$, transverse and torsional displacements appear without the increase of the compressive force. This phenomena is exactly the same as that of the in-plane buckling of a column.

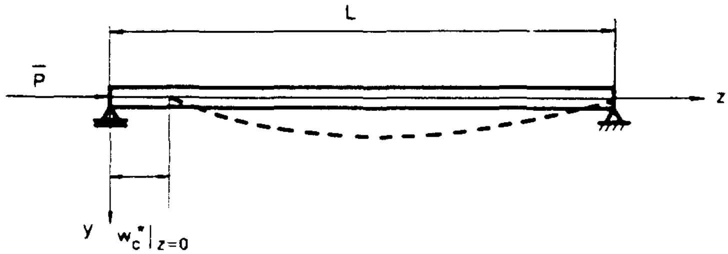

Fig. 2. Centrally compressed column.

4.2. *Internal strain energy*

As explained in Section 4.1, there are no out-of-plane displacements before buckling and no increment in axial force at buckling. Hence, the strain energy expressed in terms of incremental displacements up to the second order terms is given as follows, simplified with the help of eqns (4)–(6).

$$U = \frac{1}{2}\int_0^L \{EI_x(u_c^{*\prime\prime})^2 + EI_y(v_c^{*\prime\prime})^2 + GJ(\theta^{*\prime})^2 + EI_\omega(\theta^{*\prime\prime})^2\}\,dz. \tag{9}$$

This expression shown by Bleich (1952) is the same as Ojalvo (1987b) used.

4.3. *Potential energy of external force*

If the axial displacement of this structure is fixed at $z = L$ as shown in Fig. 2, the potential energy of external force is given by

$$W = -\bar{P}w_c^*|_{z=0} \tag{10}$$

where $w_c^*|_{z=0}$ is the incremental axial displacement of centroid at $z = 0$. $w_c^*|_{z=0}$ is further to be expressed in terms of the incremental out-of-plane displacements up to the second order terms.

Consider an infinitesimal fiber dz of the column initially parallel to the centroidal axis as shown in Fig. 2. The length of the fiber just before buckling and that after buckling are denoted here respectively by $(1+\bar{e}_z)\,dz$ and $(1+\bar{e}_z+e_z^*)\,dz$.

As easily seen from Fig. 3, these quantities can be expressed by displacement components as

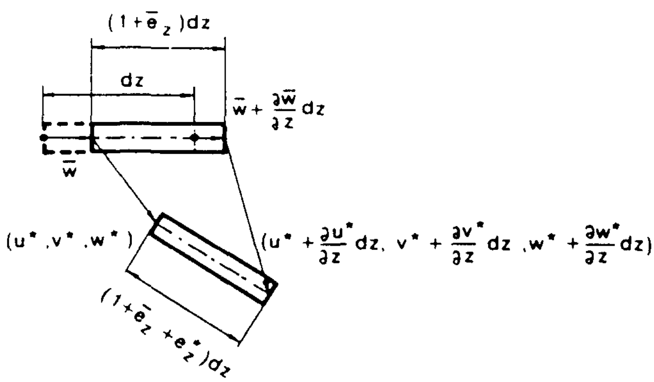

Fig. 3. Deformation of infinitesimal longitudinal element.

$$(1+\bar{e}_z)\,\mathrm{d}z = \left(1+\frac{\partial \bar{w}}{\partial z}\right)\mathrm{d}z$$

$$(1+\bar{e}_z+e_z^*)\,\mathrm{d}z = \left\{\sqrt{\left[\left(1+\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)^2+\left(\frac{\partial u^*}{\partial z}\right)^2+\left(\frac{\partial v^*}{\partial z}\right)^2\right]}\right\}\mathrm{d}z \qquad (11a,b)$$

where $\bar{e}_z$ is interpreted as an axial strain before buckling and e_z^* is an increment of it at buckling.

Since the displacements are small, the following condition holds.

$$\frac{\partial u^*}{\partial z} \ll 1, \quad \frac{\partial v^*}{\partial z} \ll 1, \quad \frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z} \ll 1. \qquad (12a–c)$$

From eqn (11b), approximated by eqns (12), $\bar{e}_z+e_z^*$ is expressed in terms of incremental displacement up to the second order terms as

$$\bar{e}_z+e_z^* = \frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}+\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2. \qquad (13)$$

The validity of the above approximation procedure is explained in the Appendix.

Considering eqns (11a) and (13), the incremental strain e_z^* is given by

$$e_z^* = \frac{\partial w^*}{\partial z}+\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2. \qquad (14)$$

Introducing eqns (3) into eqn (14), e_z^* can be rewritten in the form

$$e_z^* = w_c^{*\prime}-xu_s^{*\prime\prime}-yv_s^{*\prime\prime}-\omega\theta^{*\prime\prime}+\tfrac{1}{2}\{u_s^{*\prime}-(y-y_s)\theta^{*\prime}\}^2+\tfrac{1}{2}\{v_s^{*\prime}+(x-x_s)\theta^{*\prime}\}^2. \qquad (15)$$

During buckling there is no increment in axial force and, hence, the following equation holds.

$$\int_A \sigma_z^*\,\mathrm{d}A = E\int_A e_z^*\,\mathrm{d}A = 0 \qquad (16)$$

where σ_z^* is an increment of axial stress.

Substituting eqn (15) into eqn (16) and considering eqns (4) and (5) yields

$$Aw_c^{*\prime}+\tfrac{1}{2}A\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\}+A(y_s\theta^{*\prime}u_s^{*\prime}-x_s\theta^{*\prime}v_s^{*\prime}) + \frac{1}{2}\int_A\{(y-y_s)^2+(x-x_s)^2\}\,\mathrm{d}A(\theta^{*\prime})^2 = 0. \qquad (17)$$

Equation (17) can be solved for $w_c^{*\prime}$ as

$$w_c^{*\prime} = -\tfrac{1}{2}\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\}-y_s\theta^{*\prime}u_s^{*\prime}+x_s\theta^{*\prime}v_s^{*\prime} - \frac{1}{2A}\int_A\{(y-y_s)^2+(x-x_s)^2\}\,\mathrm{d}A\,(\theta^{*\prime})^2. \qquad (18)$$

Axial displacement $w_c^*|_{z=0}$ at $z = 0$ can finally be obtained as follows, by integrating eqn (18) with respect to z and then introducing the boundary condition that the axial displacement is fixed at $z = L$.

$$w_c^*|_{z=0} = \frac{1}{2}\int_0^L \left[(u_s^{*\prime})^2 + (v_s^{*\prime})^2 + 2\theta^{*\prime}(y_s u_s^{*\prime} - x_s v_s^{*\prime}) + \underline{\frac{1}{A}\int_A \{(y-y_s)^2 + (x-x_s)^2\}\,\mathrm{d}A\,(\theta^{*\prime})^2} \right] \mathrm{d}z. \tag{19}$$

Wagner effect results from the underlined term.

4.4. *Governing differential equations*

Substituting eqns (9), (10) and (19) into eqn (8) and integrating by parts lead to

$$\begin{aligned}
\delta U + \delta W = [& -\{EI_y u_s^{*(3)} + \bar{P}(u_s^{*\prime} + y_s\theta^{*\prime})\}\,\delta u_s^* \\
& -\{EI_x v_s^{*(3)} + \bar{P}(v_s^{*\prime} - x_s\theta^{*\prime})\}\,\delta v_s^* \\
& -\{EI_\omega \theta^{*(3)} - GJ\theta^{*\prime} + \bar{P}(y_s u_s^{*\prime} - x_s v_s^{*\prime} + r_p^2\theta^{*\prime})\}\,\delta\theta^* \\
& + EI_y u_s^{*\prime\prime}\,\delta u_s^{*\prime} + EI_x v_s^{*\prime\prime}\,\delta v_s^{*\prime} + EI_\omega \theta^{*\prime\prime}\,\delta\theta^{*\prime}]_0^L \\
& + \int_0^L [\{EI_y u_s^{*(4)} + \bar{P}(u_s^{*\prime} + y_s\theta^{*\prime})^\prime\}\,\delta u_s^* \\
& + \{EI_x v_s^{*(4)} + \bar{P}(v_s^{*\prime} - x_s\theta^{*\prime})^\prime\}\,\delta v_s^* \\
& + \{EI_\omega \theta^{*(4)} - GJ\theta^{*\prime\prime} + \bar{P}(y_s u_s^{*\prime\prime} - x_s v_s^{*\prime\prime} + r_p^2\theta^{*\prime\prime})\}\,\delta\theta^*]\,\mathrm{d}z = 0
\end{aligned} \tag{20}$$

where

$$r_p = \sqrt{\left[\frac{1}{A}\int_A \{(y-y_s)^2 + (x-x_s)^2\}\,\mathrm{d}A\right]}. \tag{21}$$

From geometrical boundary conditions at $z = 0$ and L, virtual displacements have to satisfy the equations:

$$\delta u_s^* = \delta v_s^* = \delta\theta^* = 0 \quad \text{at} \quad z = 0, L. \tag{22}$$

Thus, the necessary and sufficient conditions for eqn (20) to hold for any virtual displacement, yield the following mechanical boundary conditions and equilibrium equations.

(Mechanical boundary conditions)

$$EI_y u_s^{*\prime\prime} = 0, \quad EI_x v_s^{*\prime\prime} = 0, \quad EI_\omega \theta^{*\prime\prime} = 0, \quad \text{at} \quad z = 0, L. \tag{23a-c}$$

(Equilibrium equations)

$$\left.\begin{aligned} &EI_y u_s^{*(4)} + \bar{P}(u_s^{*\prime} + y_s\theta^{*\prime})' = 0 \\ &EI_x v_s^{*(4)} + \bar{P}(v_s^{*\prime} - x_s\theta^{*\prime})' = 0 \\ &EI_\omega \theta^{*(4)} - GJ\theta^{*\prime\prime} + \bar{P}(y_s u_s^{*\prime\prime} - x_s v_s^{*\prime\prime} + r_p^2\theta^{*\prime\prime}) = 0 \end{aligned}\right\}. \tag{24a–c}$$

5. BUCKLING OF A MONOSYMMETRIC BEAM UNDER END EQUAL MOMENTS

5.1. *Buckling phenomena*

We consider a simply supported thin-walled monosymmetric beam of length L under end equal moments. As shown in Fig. 4, coordinates are so chosen that the y axis coincides with the symmetric axis of the cross section of the beam.

Up to the critical bending moment $\bar{M}_{xcr}$, this beam deforms in the y–z plane. Similar to the buckling of a column, buckling phenomena of this structure is classified as a symmetric bifurcation. Thus, when the bending moment $\bar{M}_x$ reaches the critical value $\bar{M}_{xcr}$, the out-of-plane displacements appear without the increase of the bending moment.

5.2. *Internal strain energy*

It is well known that in-plane deformation before buckling has an effect on the critical moment $\bar{M}_{xcr}$ (Nishino *et al.*, 1973; Trahair and Woolcock, 1973). Therefore, in an exact formulation, it is necessary to consider this effect in the calculation of strain energy. However, the effect of the pre-buckling deformation on the usual structures is small in magnitude (Nethercot, 1983) and this effect is generally ignored in practical analysis. Thus, following the general practice as Ojalvo did in his formulation (1987b), we here ignore the deformation before buckling and adopt the internal strain energy given by eqn (9).

5.3. *Potential energy of external force*

Under equal end moments, the external potential energy of the simply supported beam is given by

$$W = -\bar{M}_x(v_s^{*\prime}|_{z=0} - v_s^{*\prime}|_{z=L}). \tag{25}$$

The above rotational displacements are further to be expressed by the incremental out-of-plane displacements up to the second order terms. It should be noted that $v^{*\prime}$ on the y axis at both ends of the beam have the same value, that is $v^{*\prime} = v_s^{*\prime}$, since torsional angle θ^* is zero due to the geometrical boundary conditions of the simply supported beam. Thus, eqn (25) holds regardless of the location of the applied moment, as long as it is applied on the y axis.

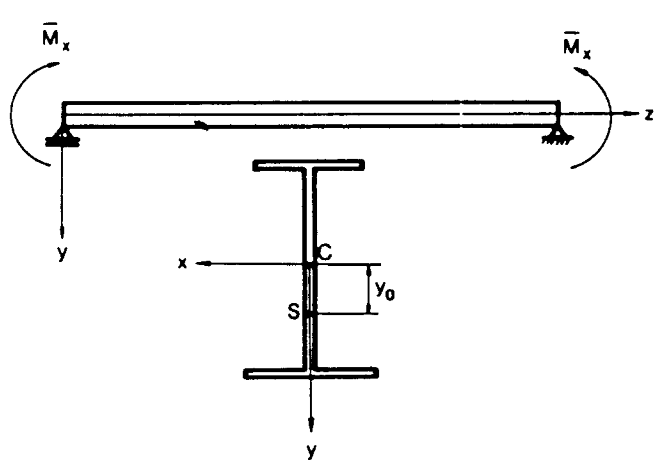

Fig. 4. Monosymmetric beam under equal end moments.

Consider an infinitesimal fiber dz of a beam initially parallel to the centroidal axis. The length of the fiber just before buckling and that after buckling are respectively given as follows, in the similar manner as shown in eqns (11).

$$(1+\bar{e}_z)\,dz = \left\{\sqrt{\left[\left(1+\frac{\partial \bar{w}}{\partial z}\right)^2+\left(\frac{\partial \bar{v}}{\partial z}\right)^2\right]}\right\}dz$$

$$(1+\bar{e}_z+e_z^*)\,dz = \left\{\sqrt{\left[\left(1+\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)^2+\left(\frac{\partial u^*}{\partial z}\right)^2+\left(\frac{\partial \bar{v}}{\partial z}+\frac{\partial v^*}{\partial z}\right)^2\right]}\right\}dz. \quad (26a, b)$$

It should be noticed that eqns (26) include the in-plane transverse displacement $\bar{v}$ before buckling, different from eqn (11).

Making use of the condition of small displacements expressed by

$$\frac{\partial u^*}{\partial z} \ll 1, \quad \frac{\partial \bar{v}}{\partial z}+\frac{\partial v^*}{\partial z} \ll 1, \quad \frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z} \ll 1 \quad (27a\text{–}c)$$

eqns (26) are approximated as

$$\bar{e}_z = \frac{\partial \bar{w}}{\partial z}+\frac{1}{2}\left(\frac{\partial \bar{v}}{\partial z}\right)^2$$

$$\bar{e}_z+e_z^* = \frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}+\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial \bar{v}}{\partial z}+\frac{\partial v^*}{\partial z}\right)^2. \quad (28a, b)$$

The above approximation procedure is exactly the same as that explained previously in Section 4.3 and the Appendix. Thus, the incremental strain e_z^* is finally given by

$$e_z^* = \frac{\partial w^*}{\partial z}+\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2+\frac{\partial \bar{v}}{\partial z}\frac{\partial v^*}{\partial z}. \quad (29)$$

Since there are no out-of-plane displacements before buckling, the transverse displacement $\bar{v}$ coincides with that on the shear center $\bar{v}_s$. Substituting the incremental displacement field of eqns (3) into eqn (29), eqn (29) is rewritten as follows.

$$e_z^* = w_c^{*\prime}-xu_s^{*\prime\prime}-yv_s^{*\prime\prime}-\omega\theta_s^{*\prime\prime}+\tfrac{1}{2}\{u_s^{*\prime}-(y-y_s)\theta^{*\prime}\}^2$$
$$+\tfrac{1}{2}\{v_s^{*\prime}+(x-x_s)\theta^{*\prime}\}^2+\bar{v}_s^{\prime}v_s^{*\prime}+(x-x_s)\bar{v}_s^{\prime}\theta_s^{*\prime}. \quad (30)$$

Due to monosymmetry of the cross section, eqn (30) is further simplified as

$$e_z^* = w_c^{*\prime}-xu_s^{*\prime\prime}-yv_s^{*\prime\prime}-\omega\theta_s^{*\prime\prime}+\tfrac{1}{2}(u_s^{*\prime})^2+\tfrac{1}{2}(v_s^{*\prime})^2$$
$$-(y-y_s)u_s^{*\prime}\theta^{*\prime}+xv_s^{*\prime}\theta^{*\prime}+\tfrac{1}{2}\{(y-y_s)^2+x^2\}(\theta^{*\prime})^2+\bar{v}_s^{\prime}v_s^{*\prime}+x\bar{v}_s^{\prime}\theta^{*\prime}. \quad (31)$$

As explained in Section 5.1, there is no increment in applied end moment $\bar{M}_x$ at buckling. The following equation holds.

$$\Delta\bar{M}_x = \int_A \sigma_z^* y\,dA = E\int_A e_z^* y\,dA \equiv 0. \quad (32)$$

Substituting eqn (31) into eqn (32) and considering eqns (5) and (6), eqn (32) yields

$$-I_x v_s^{*\prime\prime} - I_x u_s^{*\prime}\theta^{*\prime} + \tfrac{1}{2} I_x \beta_x (\theta^{*\prime})^2 = 0 \tag{33}$$

where

$$\beta_x = \frac{1}{I_x}\int_A (x^2 y + y^3)\,\mathrm{d}A - 2y_s. \tag{34}$$

It should be noted in eqn (33) that the effect of pre-buckling displacement disappears in the course of integration over the cross-sectional area without ignoring it.

Equation (33) can be integrated with respect to z as

$$v_s^{*\prime}|_{z=0} - v_s^{*\prime}|_{z=L} = \int_0^L [u_s^{*\prime}\theta^{*\prime} - \tfrac{1}{2}\beta_x(\theta^{*\prime})^2]\,\mathrm{d}z. \tag{35}$$

Substitution of eqn (35) into eqn (25) finally gives the potential energy of external force:

$$W = -\bar{M}_x \int_0^L \{u_s^{*\prime}\theta^{*\prime} - \tfrac{1}{2}\beta_x(\theta^{*\prime})^2\}\,\mathrm{d}z. \tag{36}$$

Equation (36) is rewritten as follows, integrating by parts and considering the geometrical boundary conditions at both ends.

$$W = \bar{M}_x \int_0^L \{u_s^{*\prime\prime}\theta^{*} + \tfrac{1}{2}\beta_x(\theta^{*\prime})^2\}\,\mathrm{d}z. \tag{37}$$

The above expression exactly coincides with that shown by Kitipornchai *et al.* (1986). Substituting eqns (9) and (36) into eqn (8) and integrating by parts lead to

$$\begin{aligned}\delta\Pi + \delta W = {} & [EI_y u_s^{*\prime\prime}\,\delta u_s^{*\prime} + EI_x v_s^{*\prime\prime}\,\delta v_s^{*\prime} + EI_\omega \theta^{*\prime\prime}\,\delta\theta^{*\prime} \\ & + (-EI_\omega \theta^{*(3)} + GJ\theta^{*\prime} + \bar{M}_x\beta_x\theta^{*\prime} - \bar{M}_x u_s^{*\prime})\,\delta\theta^{*} \\ & + (-EI_y u_s^{*(3)} - \bar{M}_x\theta^{*\prime})\,\delta u_s^{*} - EI_x v_s^{*(3)}\,\delta v_s^{*}]_0^L \\ & + \int_0^L [(EI_y u_s^{*(4)} + \bar{M}_x\theta^{*\prime\prime})\,\delta u_s^{*} + EI_x v_s^{*(4)}\,\delta v_s^{*} \\ & + \{EI_\omega\theta^{*(4)} - (GJ + \bar{M}_x\beta_x)\theta^{*\prime\prime} + \bar{M}_x u_s^{*\prime\prime}\}\,\delta\theta^{*}]\,\mathrm{d}z = 0.\end{aligned} \tag{38}$$

Under the geometrical boundary conditions of eqns (22), eqn (38) yields the following mechanical boundary conditions and equilibrium equations.

(Mechanical boundary conditions)

$$EI_y u_s^{*\prime\prime} = 0, \quad EI_x v_s^{*\prime\prime} = 0, \quad EI_\omega \theta^{*\prime\prime} = 0 \quad \text{at} \quad z = 0, L. \tag{39a–c}$$

(Equilibrium equations)

$$\left.\begin{aligned} & EI_y u_s^{*(4)} + \bar{M}_x\theta^{*\prime\prime} = 0 \\ & EI_x v_s^{*(4)} = 0 \\ & EI_\omega\theta^{*(4)} - (GJ + \bar{M}_x\beta_x)\theta^{*\prime\prime} + \bar{M}_x u_s^{*\prime\prime} = 0 \end{aligned}\right\}. \tag{40a–c}$$

With our consistent formulation, the theory including Wagner effect is naturally obtained also for a monosymmetric beam under end equal moments.

6. DISCUSSION

6.1. *Buckling of a column*

Ojalvo's formulation is different from the present author's only in the derivation of the axial displacement $w_c^*|_{z=0}$ at centroid, which is shown in Section 4.3. In the derivation, he assumes the incompressibility of the centroidal axis at buckling. His procedure can be explained using the equations shown in this paper.

The condition of the incompressible centroidal axis at buckling is expressed mathematically as

$$e_z^* = 0 \quad \text{at} \quad (x = 0, y = 0). \tag{41}$$

Substituting eqn (41) into eqn (15), eqn (15) can be solved for $w_c^{*\prime}$ as

$$w_c^{*\prime} = -\tfrac{1}{2}\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\} - y_s\theta^{*\prime}u_s^{*\prime} + x_s\theta^{*\prime}v_s^{*\prime} - \tfrac{1}{2}(x_s^2+y_s^2)(\theta^{*\prime})^2. \tag{42}$$

$w_c^*|_{z=0}$ can be obtained as follows by integrating eqn (42) in the same manner as was used in deriving eqn (19).

$$w_c^*|_{z=0} = \frac{1}{2}\int_0^L \{(u_s^{*\prime})^2+(v_s^{*\prime})^2+2\theta^{*\prime}(y_s u_s^{*\prime}-x_s v_s^{*\prime})+(x_s^2+y_s^2)(\theta^{*\prime})^2\}\, \mathrm{d}z. \tag{43}$$

The above is what was obtained by Ojalvo (1987b).

With this axial displacement, he derived the same governing equations as he presented in his original paper (Ojalvo, 1981), possibly following the procedure described in Section 4.4.

However, the use of eqn (43) results in a change of axial force during buckling. This is shown in the following.

Substituting eqn (42) into eqn (15), eqn (15) is reduced to

$$e_z^* = -xu_s^{*\prime\prime} - yv_s^{*\prime\prime} - \omega\theta^{*\prime\prime} - (yu_s^{*\prime} - xv_s^{*\prime})\theta^{*\prime} - (yy_s^* + xx_s^*)(\theta^{*\prime})^2 + \tfrac{1}{2}(x^2+y^2)(\theta^{*\prime})^2. \tag{44}$$

Increment of axial force $\Delta\bar{P}$ during buckling is calculated as follows using eqn (44).

$$\Delta\bar{P} = \int_A \sigma_z^*\, \mathrm{d}A = E\int_A e_z^*\, \mathrm{d}A = \tfrac{1}{2}E\int_A (x^2+y^2)\, \mathrm{d}A\ (\theta^{*\prime})^2. \tag{45}$$

It can be confirmed from eqn (45) that the magnitude of axial force $\bar{P}$ changes during buckling. Thus, Ojalvo's assumption does not agree with the buckling phenomena of the present column and leads to erroneous results (Ojalvo, 1981, 1987b).

Bleich (1952) presented the total potential energy of a column under compressive force uniformly distributed at both ends of the column. When deriving external potential energy, he considered the compressive deformation of member axis at buckling.

However, Bleich showed that the potential energy considering the deformation of the member axis can be exactly reduced to the one assuming the incompressible member axis, if the condition of no change of external force is introduced. Thus, as far as the column under uniformly distributed load is concerned, the assumption of incompressible member axis happens to yield correct results. Nevertheless, the assumption itself is incorrect, which will be explained in the following, utilizing the equations in this paper.

If we do not assume the incompressible member axis, the derivative of axial displacement at an arbitrary point on the cross section is given by

$$\frac{\partial w^*}{\partial z} = -e_z^* - \frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2. \tag{46}$$

Substituting eqns (3a, b), (15) and (18) into eqn (46), eqn (46) is reduced to

$$\begin{aligned}\frac{\partial w^*}{\partial z} = &-\tfrac{1}{2}\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\}+u_s^{*\prime}\theta^{*\prime}(y-y_s)-v_s^{*\prime}\theta^{*\prime}(x-x_s)\\ &-\tfrac{1}{2}\{(y-y_s)^2+(x-x_s)^2\}(\theta^{*\prime})^2\\ &+xu_s^{*\prime\prime}+yv_s^{*\prime\prime}+\omega\theta^{*\prime\prime}+(yu_s^{*\prime}-xv_s^{*\prime})\theta^{*\prime}\\ &-\tfrac{1}{2}\{(y-y_s)^2+(x-x_s)^2\}(\theta^{*\prime})^2\\ &+\frac{1}{2A}\int_A\{(y-y_s)^2+(x-x_s)^2\}\,\mathrm{d}A\;(\theta^{*\prime})^2.\end{aligned} \tag{47}$$

By integration of eqn (47), the shortening δ^* of the longitudinal fiber is given by

$$\begin{aligned}\delta^* = \frac{1}{2}\int_0^L\Bigg[&\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\}-2\theta^{*\prime}\{(y-y_s)u_s^{*\prime}-(x-x_s)v_s^{*\prime}\}\\ &+\{(y-y_s)^2+(x-x_s)^2\}(\theta^{*\prime})^2\\ &\underline{-2xu_s^{*\prime\prime}-2yv_s^{*\prime\prime}-2\omega\theta^{*\prime\prime}-2(yu_s^{*\prime}-xv_s^{*\prime})\theta^{*\prime}}\\ &\underline{+\{(y-y_s)^2+(x-x_s)^2\}(\theta^{*\prime})^2}\\ &\underline{-\frac{1}{A}\int_A\{(y-y_s)^2+(x-x_s)^2\}\,\mathrm{d}A\;(\theta^{*\prime})^2}\Bigg]\,\mathrm{d}z.\end{aligned} \tag{48}$$

The underlined terms result from the compressive or the extensional deformation of the member axis, while the others are the same as those derived assuming an incompressible member axis, i.e. $e_z^* = 0$.

With the axial shortening given by eqn (48), we can calculate, as follows, the external potential energy under uniformly distributed compressive force $\bar{\sigma}$.

$$W = -\int_A \bar{\sigma}\delta^*\,\mathrm{d}A = -\bar{\sigma}\int_A \delta^*\,\mathrm{d}A. \tag{49}$$

If eqn (48) is substituted into eqn (49), all the underlined terms in eqn (48) disappear in the course of integration over the cross-sectional area from eqns (4) and (5a, b). As a result, the external potential energy under uniformly distributed load exactly coincides with that derived with the assumption of incompressible member axis, as given by

$$\begin{aligned}W = -\tfrac{1}{2}\bar{\sigma}A\int_0^L\Bigg[&(u_s^{*\prime})^2+(v_s^{*\prime})^2+2\theta^{*\prime}(y_su_s^{*\prime}-x_sv_s^{*\prime})\\ &+\frac{1}{A}\int_A\{(y-y_s)^2+(x-x_s)^2\}\,\mathrm{d}A\;(\theta^{*\prime})^2\Bigg]\,\mathrm{d}z.\end{aligned} \tag{50}$$

Considering $\bar{\sigma}A = \bar{P}$, eqn (50) further coincides with the external potential energy of a column under concentrated compressive force which is expressed by eqns (10) and (19). From the above derivation, we can understand that the assumption of incompressible member axis happens to yield correct results in Bleich's case. However, it is easily verified, as shown below, that the assumption itself is incorrect even in this case.

If the member axis is incompressible during buckling, the incremental axial strain e_z^* is zero over the cross-sectional area. This means that e_z^* is always zero regardless of the values of x, y and ω. In order for this condition to be satisfied, the following equations hold from eqn (15).

$$w_c^{*\prime}+\tfrac{1}{2}\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\} = \theta^{*\prime} = u_s^{*\prime\prime} = v_s^{*\prime\prime} = 0. \tag{51}$$

Considering the end conditions, eqns (51) lead to no incremental transverse displacements during buckling, which undoubtedly violates the buckling phenomena of the column.

6.2. *Buckling of a beam*

Under end equal moments, external potential energy given by Ojalvo (1987a, b) is reduced to

$$W = \bar{M}_x \int_0^L [\theta^*(u_s^{*\prime\prime}+y_s\theta^{*\prime\prime})+\tfrac{1}{2}y_s(\theta^{*\prime})^2]\,\mathrm{d}z. \tag{52}$$

To demonstrate the discrepancy between eqn (36) and eqn (52), eqn (52) can be rewritten as follows, integrating by parts and substituting the geometrical boundary conditions at both ends.

$$W = -\bar{M}_x \int_0^L [u_s^{*\prime}\theta^{*\prime}+\tfrac{1}{2}y_s(\theta^{*\prime})^2]\,\mathrm{d}z. \tag{53}$$

Considering that W is originally expressed by

$$W = \bar{M}_x \int_0^L v_s^{*\prime\prime}\,\mathrm{d}z \tag{54}$$

$v_s^{*\prime\prime}$ can be given from eqn (53) as

$$v_s^{*\prime\prime} = -u_s^{*\prime}\theta^{*\prime}-\tfrac{1}{2}y_s(\theta^{*\prime})^2. \tag{55}$$

The above equation is interpreted as what is used by Ojalvo to express $v_s^{*\prime\prime}$ instead of eqn (33). However, the use of eqn (55) results in an increment of the external moment during buckling, which is shown in the following.

Substituting eqn (55) into eqn (31), axial strain e_z^* is reduced to

$$e_z^* = w_c^{*\prime}-xu_s^{*\prime\prime}-\omega\theta^{*\prime\prime}+\tfrac{1}{2}\{(u_s^{*\prime})^2+(v_s^{*\prime})^2\}+y_su_s^{*\prime}\theta^{*\prime}+xv_s^{*\prime}\theta^{*\prime}$$
$$+\tfrac{1}{2}(y^2-yy_s+y_s^2+x^2)(\theta^{*\prime})^2+\bar{v}_s v_s^{*\prime}+x\bar{v}_s\theta^{*\prime}. \tag{56}$$

Increment of the applied moment at buckling can be calculated as

$$\Delta\bar{M}_x = \int_A \sigma_z^* y\,\mathrm{d}A = E\int_A e_z^* y\,\mathrm{d}A = \tfrac{1}{2}EI_x(\beta_x+y_s)(\theta^{*\prime})^2. \tag{57}$$

It can be confirmed from eqn (57) that the magnitude of external moment $\bar{M}_x$ increases during buckling. Thus, as in the case of the column, Ojalvo's buckling theory can not correctly explain the buckling phenomena of the beam.

7. CONCLUDING REMARKS

In order to show the validity of the Wagner hypothesis, the governing equations for the flexural–torsional buckling of a column as well as that of a beam are derived based on the theorem of minimum potential energy without using Wagner hypothesis itself.

When deriving the potential energy of external forces, it is necessary to express the displacements of external forces in terms of the incremental displacements whose components newly appear at buckling. As known from the elastic buckling phenomena of a column and a beam, there is no change of the external forces during buckling. Hence, it is proper to use the above assumption in the calculation of the incremental displacements of external forces. Following this procedure, authors naturally obtained governing equations including the Wagner effect.

On the other hand, Ojalvo obtained new buckling theories of a column and a beam, using assumptions different from authors'. That is, in the derivation of the column theory, he adopted the assumption of incompressible member axis to calculate incremental axial displacements. In the beam theory, he derived external potential energy from a geometrical consideration on a deformed configuration of a beam without introducing any specific assumptions. However, both of Ojalvo's theories result in a change of external forces during buckling, thus violating the symmetric buckling phenomena of a column and a beam. Since there is no guarantee in the theories that will always agree with the buckling phenomena, it can be concluded that Ojalvo's theories are incorrect.

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APPENDIX

Approximation of eqn (11b)

Taylor expansion of eqn (11b) gives the following expression for strain:

$$\bar{e}_z + e_z^* = \tfrac{1}{2}\gamma - \tfrac{1}{8}\gamma^2 + O(\gamma^3) = \tfrac{1}{2}\gamma\{1 - \tfrac{1}{4}\gamma + O(\gamma^2)\} \tag{A1}$$

$$\gamma = 2\left(\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)+\left(\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)^2+\left(\frac{\partial u^*}{\partial z}\right)^2+\left(\frac{\partial v^*}{\partial z}\right)^2. \tag{A2}$$

From eqns (12), γ is negligibly small compared with unity, that is

$$\gamma \ll 1. \tag{A3}$$

Since the nonlinear terms of γ in eqn (A1) can be ignored with the help of eqn (A3), eqn (A1) is approximated as

$$\bar{e}_z+e_z^* = \tfrac{1}{2}\gamma = \frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}+\frac{1}{2}\left(\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2. \tag{A4}$$

Equation (A4) can be rewritten as

$$\bar{e}_z+e_z^* = \left(\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)\left\{1+\frac{1}{2}\left(\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z}\right)\right\}+\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2. \tag{A5}$$

Considering eqn (12c), $\frac{1}{2}[(\partial \bar{w}/\partial z)+(\partial w^*/\partial z)]$ can further be ignored compared with unity, thus resulting in eqn (13). The rest of the terms, i.e. $\frac{1}{2}(\partial u^*/\partial z)^2$ and $\frac{1}{2}(\partial v^*/\partial z)^2$, cannot be ignored, because there is no such quantity that can be compared to confirm that these terms are negligibly small.

There is another way to derive eqn (13) from eqn (A4). Equation (A4) can be solved for $[(\partial \bar{w}/\partial z)+(\partial w^*/\partial z)]$ as

$$\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z} = -1+\sqrt{\left[1-\left\{\left(\frac{\partial u^*}{\partial z}\right)^2+\left(\frac{\partial v^*}{\partial z}\right)^2-2(\bar{e}_z+e_z^*)\right\}\right]}. \tag{A6}$$

Making use of the Taylor expansion similar to eqn (A1), eqn (A6) is expressed by

$$\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z} = \tfrac{1}{2}\lambda-\tfrac{1}{8}\lambda^2+O(\lambda^3) = \tfrac{1}{2}\lambda\{1-\tfrac{1}{4}\lambda+O(\lambda^2)\} \tag{A7}$$

$$\lambda = -\left\{\left(\frac{\partial u^*}{\partial z}\right)^2+\left(\frac{\partial v^*}{\partial z}\right)^2-2(\bar{e}_z+e_z^*)\right\}. \tag{A8}$$

Taking into account eqns (12a, b) along with the fact that the axial strain $\bar{e}_z+e_z^*$ is negligibly small compared with unity, λ can be treated as a small quantity which satisfies the following condition:

$$\lambda \ll 1. \tag{A9}$$

Thus, eqn (A7) is reduced to

$$\frac{\partial \bar{w}}{\partial z}+\frac{\partial w^*}{\partial z} = \tfrac{1}{2}\lambda = -\left\{\frac{1}{2}\left(\frac{\partial u^*}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v^*}{\partial z}\right)^2-(\bar{e}_z+e_z^*)\right\}. \tag{A10}$$

It is clear that eqn (A10) yields eqn (13).